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Phase transitions on fractals: II. Sierpinski gaskets

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Abstract. This is the second in a three paper series on phase transitions on fractals. Here we construct and investigate a family of fractals which are generalisations of the Sierpinski gaskets (SGs) to all Euclidean dimensionalities. These fractal lattices have a finite order of ramification, and can be considered 'marginal' between one-dimensional and higher-dimensional geometries. Physical models defined on them are exactly solvable. We argue that short-range spin models on the SG show no finite-temperature phase transitions. As examples, we solve a few spin models and study the resistor network and percolation problems on these lattices.

1. Introduction

This is the second in a series of three papers devoted to the subject of phase transitions on fractals. The first paper (Gefen *et al* 1983, hereafter referred to as I), concentrated on quasi one-dimensional (1D) fractals (Koch curves). In the third paper (Gefen *et al* 1984, hereafter referred to as III) we study phase transitions on Sierpinski carpets.

In the present paper we focus on the *Sierpinski gaskets* (SGs) and their generalisations to d Euclidean dimensions. The SGs are interesting since they are non-trivial lattices on which physical models can be exactly solved. These models exhibit critical behaviour which is intermediate between those of 1D and higher-dimensional ones. Another important motivation for studying critical phenomena on the generalised SGs is that these lattices seem to imitate in many respects the backbone of the infinite cluster at the percolation threshold (Gefen *et al* 1981b). The exact solution of various physical problems on these lattices may help us understand better the behaviour of related models on the backbone.

The outline of the paper is as follows. In § 2 we describe the construction of SGs in a general Euclidean dimensionality d. In § 3, we solve Ising models on the two-dimensional (2D) and 3D gaskets. For the 2D SG we also generalise our analysis and solve the q-state Potts model. Section 4 considers percolation on our lattices. Finally, § 5 is devoted to the solution of the DC conductivity of resistor networks on these fractals. This latter problem is probably related to the low-temperature behaviour of continuous ($n \geq 2$) spin models.

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2. Geometry

Our geometrical model consists of a generalisation of the Sierpinski gasket (Mandelbrot 1977, 1982, Gefen *et al* 1980, 1981b) to d Euclidean dimensions: one starts with a d-dimensional hypertetrahedron. The midpoints of the edges are connected, creating $(d+1)$ small hypertetrahedra. The volume at the centre (bounded by the faces of these new tetrahedra) is then removed, and the procedure is repeated for the $(d+1)$ new tetrahedra. The procedure is illustrated for $d=2$ in figure 1. One construction step for $d=3$ is shown in figure 2. The construction procedure is repeated down to a 'microscopic' length scale, a. Each step changes the length scale by a factor 2, and creates $(d+1)$ new units. Therefore the fractal dimensionality (Mandelbrot 1977, 1982) is given by

$$D=\ln(d+1)/\ln 2. \tag{2.1}$$

Figure 1. The SG in 2D. The initial triangle and the first three construction stages are shown. The limit shape has the Euclidean dimensionality $d=2$, and the fractal dimensionality $D=\ln 3/\ln 2\simeq 1.585$.

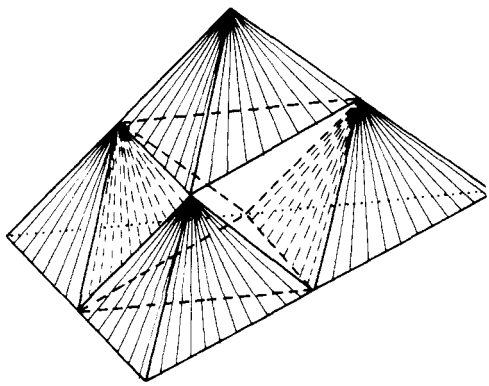

Figure 2. First construction stage for SG with $d=3$, $D=\ln 4/\ln 2=2$.

Another important geometrical feature of these lattices is the order of ramification, R.† For SG $R_{\max}=2d$ and $R_{\min}=d+1$ (for points that are not lattice points), i.e. SG are 'quasihomogeneous'. Notice that if at each construction step the eliminated volume is slightly increased (decreased) then R becomes zero (infinite). In this sense our gaskets are 'marginal', similarly to the infinite cluster (or its backbone) at percolation (Gefen *et al* 1981b).

† The order of ramification R at a point P measures the smallest number of significant interactions which one must cut in order to isolate an arbitrarily large bounded subset surrounding P. The maximum and minimum values of R obey $R_{\max}\geqslant 2R_{\min}-2$. When equality prevails, a curve is 'quasihomogeneous'. When $R_{\max}=R_{\min}$ (requiring $R=2$ or $R=\infty$), a curve is 'homogeneous' (see Mandelbrot 1982, ch 14).

3. Discrete spin systems

We now consider various spin models on the SG lattices. Each spin is put on a 'microscopic' lattice site. In principle, all spin systems on the SG are exactly solvable, due to their finite order of ramification. However, the calculation becomes more complicated as the dimensionality increases; a larger number of relevant interactions should be included, and more spin states must be integrated out at each renormalisation group (RG) step. We thus concentrate here on a few cases, the generalisation to other systems being straightforward.

Hereafter we shall limit ourselves to models with nearest-neighbour interactions (on the 'microscopic' scale) only. More generally, in all the cases we studied with finite range and strength of interactions, a finite R implies that no phase transition occurs at any finite temperature, i.e. $T_c = 0$. This result is consistent with the standard inequalities or entropy arguments (see e.g. Griffiths 1972, Thompson 1972). At any small finite temperature, the system may break into domains (or generate other excitations if the spins are continuous), gaining free energy. (The system gains an entropy which is higher than the energy needed to create the excitations.) As a consequence order is destroyed at any finite T.

3.1. Ising models in 2D SG

Classical Ising spins ($S_i = \pm 1$) are put on the 'microscopic' lattice sites of figure 1 (after the iterative decoration down to the microscopic nearest-neighbour scale a was performed). At zero field, we consider the nearest-neighbour exchange Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j + A, \tag{3.1}$$

where A is a constant. The RG equations are obtained by summing over the internal spins of all the triangles of linear size $2a$. The rescaling factor here is $b = 2$. The resulting recursion relation for $K = J/kT$ is (Gefen *et al* 1980)

$$e^{4K'} = (e^{8K} - e^{4K} + 4)/(e^{4K} + 3). \tag{3.2}$$

The only fixed points of equation (3.2) are at $K = 0, \infty$. The point $K^* = 0$ is an infinite-temperature stable fixed point. $K^* = \infty$ is a zero-temperature unstable fixed point. Near $T = 0$, equation (3.2) reduces to

$$t' = t + 4t^2 + O(t^3), \tag{3.3}$$

where $t \equiv e^{-4K}$ is chosen as the low-temperature small variable. (This is a natural generalisation of the one-dimensional $t = e^{-2K}$, since now the coordination number is 4, instead of 2). Writing the correlation length as $\xi \sim t^{-\nu}$, the coefficient of the linear term is 2^y, with $y = 1/\nu$. In our case, this coefficient is equal to unity, i.e. $y = 1/\nu = 0$. The temperature is thus a 'marginal' variable. This is probably related to the marginal character of the SG, in the sense that one may have a finite T_c by introducing small changes into its structure.

For small t, the solution of (3.3) is

$$t(l) = [4(l_0 - l)]^{-1} + O[(l_0 - l)^{-2}], \tag{3.4}$$

where $4l_0 = 1/t(0) = e^{4K}$. After l iterations, the correlation length becomes $\xi(l) = \xi/2^l$.

Iterating until $\xi(l)$ becomes of order unity, this implies that $\xi \propto 2^l \propto 2^{1/4t}$, i.e.

$$\xi \propto \exp(\tfrac{1}{4} \ln 2 \exp(4K)). \tag{3.5}$$

Since the number of spins in a volume of size ξ is ξ^D, we expect the singular free energy to behave as $F \propto \xi^{-D}$. If we wrote this as $t^{2-\alpha}$, we would conclude that $\alpha = -\infty$, in agreement with the hyperscaling law $D\nu = 2 - \alpha$ and $\nu = \infty$.

Since the transition occurs at zero temperature, the magnetisation per spin 'jumps' from zero to unity at $t = 0$. Thus, the recursion relation for the magnetic field must be of the form $h' = 2^D h$, and the magnetic exponent x is equal to D (I, Nienhuis and Nauenberg 1975). Thus, $F(t, h) = \xi^{-D} F(\xi^D h)$, and we find that $M \propto \xi^0$, $\chi \propto \xi^D$, etc.

Note that the choice $t = e^{-4K}$ was arbitrary. We could have chosen $\tilde{t} = \exp(-\frac{1}{4} \ln 2 \exp(4K))$, and found that $\tilde{t}' = 2\tilde{t}$. With this parameter, we would have $y = 1/\nu = 1$.

3.2. *Ising models on* 3D SG

We now put Ising spins on the 3D SG (figure 2). The most general Hamiltonian for nearest-neighbour interactions is

$$\mathcal{H} = -J_1 \sum_{\langle ij \rangle} S_i S_j - J_2 \sum_{\langle ijkl \rangle} S_i S_j S_k S_l + A. \tag{3.6}$$

The four-spin interaction J_2 is defined for spins placed on the vertices of a tetrahedron. Using the notation $K \equiv J_1/k_B T$ and $L \equiv J_2/k_B T$, and tracing over all the spins on the midpoints of the edges, we obtain the following RG recursion relations:

$$e^{8K'} = \frac{e^{24K+4L} + e^{-4L} + 6e^{12K} + 12e^{4K} + 3e^{-4L} + 4e^{4L} + 4e^{-4L}}{2e^{12K} + 8e^{4K} + 2e^{8K+4L} + 4e^{-4L} + 12e^{4K} + 4e^{4L} + 4e^{-4L}}, \tag{3.7a}$$

$$e^{6K'+2L'} = \frac{e^{24K+4L} + e^{-4L} + 6e^{12K} + 12e^{4K} + 3e^{-4L} + 4e^{4L} + 4e^{-4L}}{e^{18K+2L} + 8e^{6K-2L} + 3e^{10K+2L} + 24e^{-2K-2L} + 15e^{2K+2L}}. \tag{3.7b}$$

The analysis here is more complicated than in the previous case, since we now have a two-dimensional parameter space, (K, L). It is easy to see that if we start with $K \gg 1$, $L = 0$, we immediately flow to $K \cong -L \gg 1$. Thus it is useful to study the behaviour near the $T = 0$ fixed point, $K \simeq -L = \infty$. To leading order near this fixed point we can write equations (3.7) as

$$e^{-8K'} \simeq 2e^{-12K-4L}(1 + 10e^{-8K} + e^{-4K+4L} - 2e^{-12K-4L}), \tag{3.8a}$$

$$e^{-6K'-2L'} \simeq e^{-6K-2L}(1 + 2e^{-12K-4L} + 3e^{-8K}). \tag{3.8b}$$

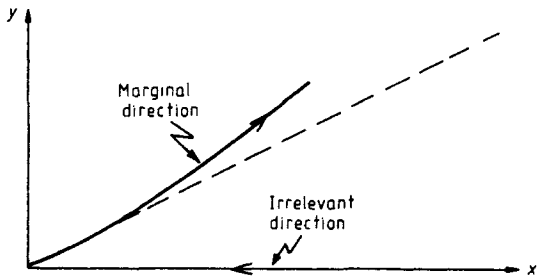

Figure 3. Schematic flow diagram in a 2D parameter space, for Ising models on 3D SG.

Taking the square of equation (3.8*b*) and using the notation $x \equiv e^{-8K}$, $y \equiv e^{-12K-4L}$ we obtain

$$x' \simeq 2y + 2x^2 + 20xy - 4y^2, \qquad y' \simeq y + 6xy + 4y^2. \tag{3.9a, b}$$

To linear order, this reduces to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.10}$$

The eigenvalues and their right eigenvectors are 0, 1 and $\binom{1}{0}$ and $\binom{2}{1}$, respectively. The eigenvalue 0 represents a highly irrelevant variable, which disappears immediately after the first iteration. The second eigenvalue, 1, is to be identified with $2^{1/\nu}$, so that again the temperature is a marginal variable and $\nu = \infty$. A general initial vector

$$\begin{pmatrix} x \\ y \end{pmatrix} = (x - 2y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{3.11}$$

will thus become

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{3.12}$$

To this order, this vector will then remain invariant, with $x(l) = 2y(l) = 2y(0)$. Again, this reflects the fact that the temperature is marginal and that $\nu = \infty$.

To linear order, the first iteration brings us to the line $x = 2y$. To find the deviations from this line, we must go to order y^2. Substituting $x = 2y + 2ay^2$ into both equations (3.9), we find that they become identical provided that we choose $a = 6$. In this case, both equations reduce to $y' = y + 16y^2$, with the solution $y(l) \simeq 1/16(l_0 - l)$, where $16l_0 \simeq y(1)^{-1} \simeq 2x(1)^{-1} \simeq 2\exp(8K)$. Combining with $\xi \propto 2^l$ we now find that

$$\xi \propto \exp(\tfrac{1}{8} \ln 2 \exp(8K)), \tag{3.13}$$

which is very similar to equation (3.5). The other thermodynamic quantities will also follow the behaviour described for the 2D case.

3.3. Potts models on 2D SG

We next study a q-state Potts model with nearest-neighbour interactions, put on the 2D SG. The most general Hamiltonian (in units of $1/\beta \equiv k_B T$) is

$$\beta\mathcal{H} = K \sum_{\langle ij \rangle} (1 - \delta_{\sigma_i \sigma_j}) + \Delta \sum_{\langle ijk \rangle} (1 - \delta_{\sigma_i \sigma_j})(1 - \delta_{\sigma_j \sigma_k})(1 - \delta_{\sigma_k \sigma_i}), \tag{3.14}$$

where σ_i (the spin attached to the ith site) may assume q different values. The three-spin interaction refers to spins sitting on the vertices of the basic ('microscopic') triangle. One RG step consists of summation over the 'internal spins', σ_1, σ_2, σ_3 while the external spins are held at fixed states, α, β, γ (see figure 4). Let us denote by $W[\overset{\alpha}{\Delta}_\gamma]$ the result of such a summation. It is convenient to use the notation $W[\overset{\alpha}{\Delta}_\alpha] \equiv D(K, q)$; $W[\overset{\alpha}{\Delta}_\beta] \equiv e^{-2K} N(K, \Delta, q)$, $\alpha \neq \beta$; $W[\overset{\alpha}{\Delta}_\gamma] \equiv M(K, \Delta, q)$, $\alpha \neq \beta \neq \gamma$. The resulting RG recursion relations are

$$e^{-2K'} = \frac{e^{-2K} N(K, \Delta, q)}{D(K, q)}, \qquad e^{-3K' - \Delta'} = \frac{M(K, \Delta, q)}{D(K, q)}. \tag{3.15a, b}$$

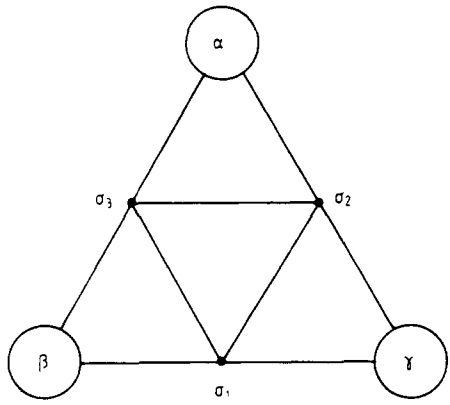

Figure 4. Potts models on 2D SG: a schematic RG step. Summation is performed over the 'internal spin', σ_1, σ_2, σ_3. α, β, γ denote states of the 'external' spins.

Since the model has no finite-temperature phase transition we are interested in the behaviour near the $T=0$ unstable fixed point ($K, \Delta \gg 1$). The other (stable) fixed point is expected to be at $T=\infty$ ($K=\Delta=0$). We thus concentrate on the low-energy excitations of D, N and M. A consistent diagrammatic calculation of these quantities is presented in the appendix. Substituting the leading terms of D, N and M in equations (3.15) we obtain

$$e^{-2K'} = e^{-2K} \frac{[1+4\,e^{-2K}+3(q-1)\,e^{-4K}+2(q-2)\,e^{-3K-\Delta}+\dots]}{[1+3(q-1)\,e^{-4K}+\dots]}, \tag{3.16a}$$

$$e^{-3K'-\Delta'} = 3\,e^{-4K} \frac{(1+2\,e^{-K-\Delta}+\dots)}{[1+3(q-1)\,e^{-4K}+\dots]}. \tag{3.16b}$$

Denoting $x \equiv e^{-2K}$, $y^2 \equiv e^{-3K-\Delta}$ we obtain the following equations to second order in x and y:

$$x' = x+4x^2, \qquad y' = \sqrt{3}x+\tfrac{1}{2}\sqrt{3}y^2. \tag{3.17a, b}$$

The parameter q enters only in the third order and affects only corrections to the critical behaviour. In the linear order equations (3.17) can be rewritten in the following way

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.18}$$

yielding the eigenvalues 0 and 1 with the eigenvectors $\binom{0}{1}$ and $\binom{1}{\sqrt{3}}$. Repeating the same procedure as before, we find that $y \simeq \sqrt{3}(x+\frac{5}{2}x^2)$. Solving the recursion relations for x then yields

$$\xi \propto \exp(\tfrac{1}{4} \ln 2 \exp(2K)), \tag{3.19}$$

independent of q. Noting that in the Ising case, $q=2$, we had $JS_iS_j \equiv J(2\delta_{S_iS_j}-1)$, this result is consistent with equation (3.5).

The examples treated here probably indicate that forms like (3.19) are typical for all the discrete spin models on general d SGs.

4. Percolation on SG

We now delete a fraction p of the bonds on the SG. Since the order of ramification is finite, we expect the SG to break into finite clusters for all $p > 0$. The point $p = 0$ is thus expected to be an unstable fixed point of the RG, for any d. The typical size of the finite clusters, ξ, diverges to infinity as $p \to 0$.

In order to construct recursion relations for the 2D SG, consider the triangle ABC in figure 5(a). On the large scale, the points B and C will be separated from the point A only if both the (renormalised) bonds AB and AC are missing (broken lines). The probability for this is $(p')^2$. On the smaller scale, A will be separated from both B and C if AD and AE are missing (figure 5(b), probability p^2), if AD, DE, EF and EC (or AE, DE, DB and DF, or DB, DF, EF and EC, or DB, BF, FC and EC, figures 5(c)–(f), all with probability p^4), etc. Thus

$$(p^2)' = p^2 + 4p^4 + O(p^5). \tag{4.1}$$

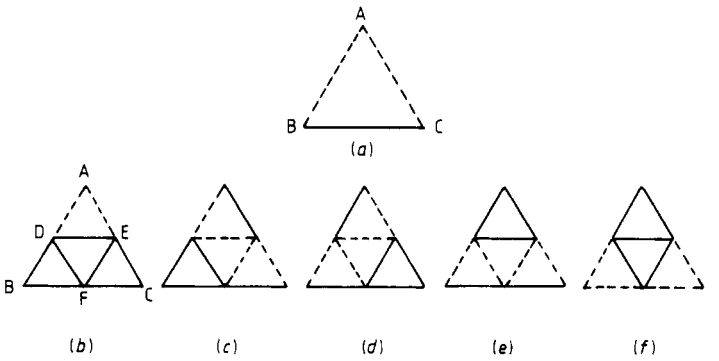

Figure 5. RG step for percolation on the 2D SG. Missing bonds are denoted by broken lines.

This is exactly the recursion relation for e^{-2K} in the Potts model, equation (3.17a). Indeed, one expects the Potts model to map on the percolation problem in the limit $q \to 1$ (e.g. Kasteleyn and Fortuin 1969), with $p \to e^{-K}$. From (3.19) we now conclude that

$$\xi \propto \exp(\tfrac{1}{4} \ln 2/p^2). \tag{4.2}$$

As before, the dependence of other quantities may be directly deduced using scaling relations.

A similar procedure for $d > 2$ also yields $(p^d)' = p^d +$ higher order, and we thus expect essential singularities like (4.2) for all SGs.

5. Electric conductance on SG

Another physical quantity which is affected by the anomalous geometry of the SGs is the electric conductance of a resistor network placed on them. Again, we approach the problem using an exact RG procedure. The magnetic bonds of the previous sections are replaced by resistors. In principle, the resistance of a large hypertetrahedron

before and after the rescaling transformation should be the same. To construct our recursion relations, we compare two hypertetrahedra of length $2a$ (a being the length of a bond before rescaling). One hypertetrahedron consists of 'bonds' (resistors) of length a, the other consists of rescaled resistors of length $2a$. We now send a current I into each of the two tetrahedra, and consequently a current I/d comes out through each of the other d corners. We require that the corresponding voltages of the two tetrahedra will be the same and find the rescaled resistance $R(2a)$ as a function of the original $R(a)$. Notice that the symmetry of the problem simplifies the calculation of currents and voltages in the system. Thus, for example, for the 2D gasket, one may send a current I into the structure shown in figure 4 through the point α, and take out currents $\frac{1}{2}I$ and $\frac{1}{2}I$ from points β and γ respectively. Using that symmetry, there will be no current flowing through the bond between σ_2 and σ_3. The equivalent resistance between α and γ will thus be

$$R_{\sigma_2\alpha} + 1/[1/R_{\sigma_2\gamma} + 1/(R_{\sigma_1\sigma_2} + R_{\sigma_1\gamma})].$$

More generally the resulting relation for the equivalent resistance is

$$R(2a) = [(d+3)/(d+1)]R(a). \tag{5.1}$$

Rewriting this as $R(2a) = 2^{\tilde{\zeta}}R(a)$ we obtain

$$\tilde{\zeta} = \ln[(d+3)/(d+1)]/\ln 2. \tag{5.2}$$

The resistance of the renormalised resistors at length scale L is thus

$$R(La) \sim L^{\tilde{\zeta}}R(a). \tag{5.3}$$

One may define the *conductivity* of an equivalent homogeneous medium at scale L as

$$\sigma(La) \sim L^{2-d}/R(La). \tag{5.4}$$

We can also write this as $\sigma(La) \sim L^{-\tilde{t}}$ with $\tilde{t} = d - 2 + \tilde{\zeta}$.

A possible by-product of these results concerns the low-temperature properties of dilute n-component spin models, with $n \geq 2$. To lowest order in $1/K = k_B T/J$, there exists an equivalence between the recursion relations for $1/K$ and for R (R being the resistivity per bond, see Stinchcombe (1979)). This equivalence also follows from the similarity between Kirchoff's equations and the equations describing spin waves at low temperatures (Kirkpatrick 1973). Equation (5.1) thus implies that for these models one has

$$T' = [(d+3)/(d+1)]T, \tag{5.5}$$

i.e.

$$y_n = 1/\nu_n = \ln[(d+3)/(d+1)]/\ln 2 = \tilde{\zeta}. \tag{5.6}$$

The correlation length is therefore predicted to diverge as $\xi \sim T^{-\nu_n}$. This power law behaviour implies that the SGS are not marginal with respect to continuous symmetry models.

6. Conclusion

In the present paper we have constructed and analysed a generalised d-dimensional family of SGS. In quite a few respects the SGS deserve special consideration. They are

non-trivial, and still exactly solvable. They can be regarded as being marginal geometries between 1D (or quasi 1D) figures and infinitely ramified fractals (e.g. the Sierpinski carpets). Employing the fact that they are exactly solvable we have solved various spin models on these lattices. We also studied their electrical conductivity and discussed the problem of percolation on these lattices.

It is straightforward to extend and generalise the above calculations to other models put on the SG. Their simplicity (and yet, non-triviality) makes them a useful geometrical realisation for various physical systems, in particular, systems at percolation (see e.g. Gefen *et al* 1981a, 1981b, Aharony *et al* 1981, Stephen 1981, Alexander 1983).

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Appendix

In this appendix we give details of the diagrammatic calculation of the leading terms in D, N and M. Table 1 summarises these leading terms. In the notation of figure 4, the first entry in the table corresponds to $\alpha = \beta = \gamma = \sigma_1 = \sigma_2 = \sigma_3$, and this common value (state) is denoted by 1. Clearly, this diagram contributes to D (as do all diagrams

Table 1. Leading contributions for *low-temperature* Potts model recursion relations.

Leading contributions to	α	β	γ	σ_1	σ_2	σ_3	Degeneracy	Energy
D	1	1	1	1	1	1	1	0
	1	1	1	2	1	1	$3(q-1)$	$4K$
	1	1	1	1	2	2	$3(q-1)$	$6K$
	1	1	1	2	2	2	$q-1$	$6K$
$e^{-2K}N$	2	1	1	1	1	1	1	$2K$
	2	1	1	1	1	2	2	$4K$
	2	1	1	1	2	2	1	$4K$
	2	1	1	2	2	2	1	$4K$
	2	1	1	2	1	1	1	$6K$
	2	1	1	2	1	2	2	$6K$
	2	1	1	3	1	1	$q-2$	$6K$
	2	1	1	1	3	3	$q-2$	$6K$
	2	1	1	3	3	3	$q-2$	$6K$
	2	1	1	1	1	3	$2(q-2)$	$5K+\Delta$
M	1	2	3	1	1	1	3	$4K$
	1	2	3	2	1	1	6	$5K+\Delta$

with $\alpha = \beta = \gamma$). Since all bonds are satisfied, the energy of this non-degenerate configuration is zero. Similarly, the second entry in the table has $\alpha = \beta = \gamma = \sigma_2 = \sigma_3 \neq \sigma_1$. There are four unsatisfied bonds, and the site σ_1 could be in any of the remaining $(q-1)$ states. The remainder of the table is self-explanatory, and the combined results yield equations (3.16).

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